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A SIMPLE METHOD OF DETERMINING STRESS IN CURVED FLEXURAL MEMBERS

BY

BENJAMIN J. WILSON

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A SIMPLE METHOD OF DETERMINING STRESS IN CURVED FLEXURAL MEMBERS*

I. Introduction

1. *Calculation of Stresses in Curved Beams.*—Curved members subjected to bending, such as crane hooks, frames for punches and drill presses, members in railway trucks, chain links, etc., are important elements in many engineering machines and structures. It is generally understood that the ordinary flexure formula, which gives reliable values for the primary stresses in *straight* beams or flexural members, gives values for stress that may be much too small when applied to *curved* flexural members. Nevertheless, this straight-beam formula is still widely used in the design of curved flexural members.

The main reasons for applying the straight-beam formula to curved beams are, probably, that the curved-beam formula is much more complicated and difficult to apply than is the straight-beam formula and that, in the case of cast frames, the deviation from the assumed condition of homogeneity of material, as well as the presence of initial stresses, causes uncertainty as to the significance of the calculated stresses. These conditions will be discussed briefly later in this circular.

Due to the shorter length of the fibers on the concave (or inner) side of a curved beam the unit strain (and hence also the unit stress within the proportional limit) is greater than that given by the straight-beam formula; and the stress on the convex (or outer) side is less. Likewise the neutral axis does not pass through the centroid of the cross-section of the beam but lies nearer to the concave or inner side.

This increase in stress due to the curvature of the beam may be of great importance, especially in members in which the stress is repeated many times, as is frequently the case.

Winkler† was perhaps the first engineer to point out clearly the difference between the stress distribution in a straight beam and in a curved beam, and to emphasize the engineering importance of the problem. Bach‡ extended the analysis and also obtained valuable experimental results. The formula resulting from this analysis, which will be referred to as the Winkler-Bach formula,§ has gained rather wide acceptance; it may be written

*The discussion herein presented applies only to members subjected to symmetrical loading, that is, where the plane of the loads contains a principal axis of inertia of each cross-section, and curved only in the plane of loading.

†For Winkler's memoir see, "Formanderung und Festigkeit Gekrummter Körper ins besondere der Ringe." Der Civilingenieur Bd. IV S. 232-246.

‡See C. Bach, "Elastizität und Festigkeit." See also "The Strength of Chain Links," Univ. of Ill. Eng. Exp. Sta. Bul. 18.

§This formula is derived in Appendix C.

$$S = \frac{M}{aR}\left[1 + \frac{1}{Z}\left(\frac{y}{R+y}\right)\right] \tag{1}$$

for pure bending; and

$$S = S_1 + \frac{M}{aR}\left[1 + \frac{1}{Z}\left(\frac{y}{R+y}\right)\right] \tag{2}$$

for combined direct stress and bending.

S = unit stress at fiber in question

M = bending moment (positive when it increases curvature, negative when it decreases curvature)

a = area of cross-section

R = distance from center of curvature to centroidal axis of section

y = distance from centroidal axis to fiber in question (negative when measured toward, and positive when measured away from center of curvature); for the extreme fiber the value of y is denoted by c as in Fig. 1.

Z = a property of the cross-section defined by the expression,

$$-\frac{1}{a}\int \frac{y}{R+y}\,da$$

S_1 = axial or direct unit stress

This formula has been shown to give results that agree well with the stresses found from experimental strain measurements of curved beams made of steel.* The equation also satisfies the limiting condition of reducing to the ordinary flexure formula when the radius of curvature becomes very large; that is, when the beam becomes practically straight.

The formula, however, although satisfactory theoretically, has a serious disadvantage because of its difficulty of application, the difficulty being in evaluating the property Z of the cross-section. The evaluation of this property of the area by an analytical method is complicated for

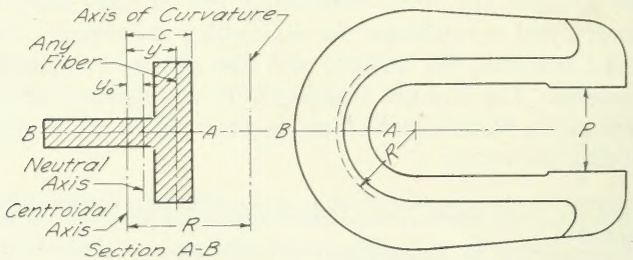


FIG. 1. A TYPICAL CURVED FLEXURAL MEMBER

*"Tests and Theory of Curved Beams," Winslow and Edmonds, A. S. M. E. paper, 1926.

all sections except the circle and the rectangle; and even for the simpler ordinary composite sections, as for instance the I or T sections, the expression becomes too complicated for quick or accurate solution.*

A possible solution of this difficulty of evaluating Z is to use graphical methods. One graphical method is the "link rigidity" method,† and another, which the writers have found to be more accurate, is the "line polygon" method.‡ Either method is long and susceptible to those errors inherent in graphical work, which in this case are serious, since Z is a small quantity for any section, rarely exceeding 0.8 for a very sharp curvature, and becoming less than 0.005 for a relatively large curvature. Experience with graphical methods of determining Z has convinced the writers that the error that may be expected is at least ten per cent, and may be considerably greater.§ Graphical methods of determining Z may be considered as a last resort when analytical expressions cannot be obtained, or are very complicated.

A way of evading these difficulties has frequently been suggested; namely, to use the straight-beam flexure formula with a correction factor. The expression for pure bending in a curved beam would then be

$$S = K\,(Mc/I) \tag{3}$$

and for combined direct stress and bending

$$S = S_1 + K\,(Mc/I) \tag{4}$$

where

$$K = \frac{\dfrac{M}{aR}\left[1 + \dfrac{1}{Z}\left(\dfrac{c}{R+c}\right)\right]}{\dfrac{Mc}{I}} \tag{5}$$

No values of K seem to be available except for circular and rectangular sections, and although it has generally been assumed that K is approximately the same for sections of all shapes, no careful study of the value of K seems to have been made.

The object of this investigation, therefore, was to determine accurately values of the correction factor K for variously shaped sections and for degrees of curvature commonly found in curved flexural members; and thus to make available the results of the curved-beam theory without involving the usual difficulties and errors encountered in determining

*See Appendix A for analytical expressions for Z.
†For description see "Strength of Materials," A. Morley, -p. 383.
‡See Appendix B.
§For a discussion of the errors likely to occur in the value of Z see "Bending Stresses in Hooks and Other Curved Beams," by A. Morley, Engineering (London) Vol. 98, Sept. 11 and 25, 1914.

the value of Z and in using the more involved curved-beam formula. The location of the neutral axis in a curved beam, a knowledge of which is sometimes of real value, is also given for various cross-sections and degrees of curvature.

2. *Acknowledgments.*—The work herein presented was originally undertaken as a part of a graduate course at the University of Illinois under Fred B. Seely, Professor of Theoretical and Applied Mechanics, and was completed under his direction. The writers gratefully acknowledge his many helpful suggestions and generous assistance.

Credit is due Mr. Albert E. Hershey, Research Assistant in Mechanical Engineering, for the graphical solution of Z and the analytical expressions for Z given in the appendices. This work was done previously as part of a graduate course under Professor Seely.

II. Correction Factor for Use With Ordinary Flexure Formula

3. *Values of Correction Factor.*—The value of the correction factor K has been computed for the inside fiber A, and the outside fiber B (Fig. 1) for 12 of the more common sections, and for 10 different degrees of curvature in each case. Every effort has been made to obtain accurate results. The values of Z were obtained from the analytical expression by use of a calculating machine, and in all cases were checked by means of graphical methods. The resulting values of K are given in Table 1. Values of K for intermediate values of $\frac{R}{c}$ may be found without appreciable error, by ordinary linear interpolation.

The range of the degree of curvature has been so chosen as to include the two extreme conditions. In the one case the curvature is so sharp as to approach the condition of localized stress, under which the mass action of the material and the distribution of stress assumed n the ordinary formulas of mechanics no longer hold;* and in the other case the curvature is so slight that the uncorrected flexure formula may be applied without serious error.

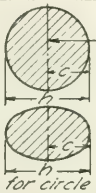
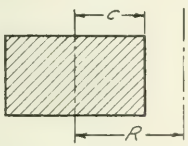
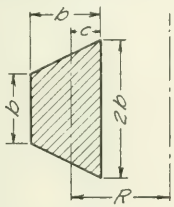
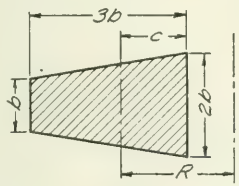
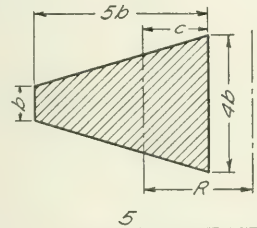
It is evident that there is considerable variation in the value of K for different sections for values of $\frac{R}{c}$ of 2.0 or less; that is, for beams in which the curvature is of real importance.

Table 1 gives values of K for sections that may be considered as typical. Fortunately there are sections for which K is independent of the breadth of the section, as noted in Table 1. For other sections K

*See F. B. Seely, "Resistance of Materials," Appendix III.

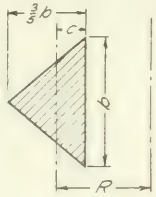
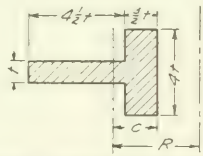
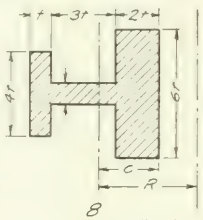
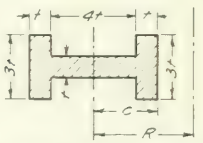
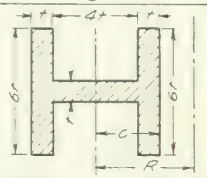
TABLE 1

VALUES OF K FOR DIFFERENT SECTIONS AND DIFFERENT RADII OF CURVATURE

Section	$\frac{R}{c}$	Factor K		y_o*
		Inside Fiber	Outside Fiber	
 K the same for circle and ellipse and independent of dimensions. 1	1.2	3.41	0.54	0.224R
	1.4	2.40	0.60	0.151R
	1.6	1.96	0.65	0.108R
	1.8	1.75	0.68	0.084R
	2.0	1.62	0.71	0.069R
	3.0	1.33	0.79	0.030R
	4.0	1.23	0.84	0.016R
	6.0	1.14	0.89	0.0070R
	8.0	1.10	0.91	0.0039R
	10.0	1.08	0.93	0.0025R
 K independent of section dimensions. 2	1.2	2.89	0.57	0.305R
	1.4	2.13	0.63	0.204R
	1.6	1.79	0.67	0.149R
	1.8	1.63	0.70	0.112R
	2.0	1.52	0.73	0.090R
	3.0	1.30	0.81	0.041R
	4.0	1.20	0.85	0.021R
	6.0	1.12	0.90	0.0093R
	8.0	1.09	0.92	0.0052R
	10.0	1.07	0.94	0.0033R
 3	1.2	3.01	0.54	0.336R
	1.4	2.18	0.60	0.229R
	1.6	1.87	0.65	0.168R
	1.8	1.69	0.68	0.128R
	2.0	1.58	0.71	0.102R
	3.0	1.33	0.80	0.046R
	4.0	1.23	0.84	0.024R
	6.0	1.13	0.88	0.011R
	8.0	1.10	0.91	0.0060R
	10.0	1.08	0.93	0.0039R
 4	1.2	3.09	0.56	0.336R
	1.4	2.25	0.62	0.229R
	1.6	1.91	0.66	0.168R
	1.8	1.73	0.70	0.128R
	2.0	1.61	0.73	0.102R
	3.0	1.37	0.81	0.046R
	4.0	1.26	0.86	0.024R
	6.0	1.17	0.91	0.011R
	8.0	1.13	0.94	0.0060R
	10.0	1.11	0.95	0.0039R
 5	1.2	3.14	0.52	0.352R
	1.4	2.29	0.54	0.243R
	1.6	1.93	0.62	0.179R
	1.8	1.74	0.65	0.138R
	2.0	1.61	0.68	0.110R
	3.0	1.34	0.76	0.050R
	4.0	1.24	0.82	0.028R
	6.0	1.15	0.87	0.012R
	8.0	1.12	0.91	0.0060R
	10.0	1.12	0.93	0.0039R

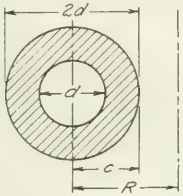
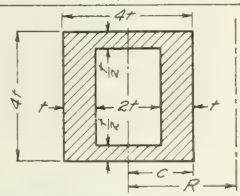
*y_o is distance from centroidal axis to neutral axis.

TABLE 1 (Continued)
VALUES OF K FOR DIFFERENT SECTIONS AND DIFFERENT RADII OF CURVATURE

Section	$\frac{R}{c}$	Factor K		y_o^*
		Inside Fiber	Outside Fiber	
 6	1.2	3.26	0.44	0.361R
	1.4	2.39	0.50	0.251R
	1.6	1.99	0.54	0.186R
	1.8	1.78	0.57	0.144R
	2.0	1.66	0.60	0.116R
	3.0	1.37	0.70	0.052R
	4.0	1.27	0.75	0.029R
	6.0	1.16	0.82	0.013R
	8.0	1.12	0.86	0.0060R
	10.0	1.09	0.88	0.0039R
 Channel section values of K same as for this section. (See Table 2) 7	1.2	3.63	0.58	0.418R
	1.4	2.54	0.63	0.299R
	1.6	2.14	0.67	0.229R
	1.8	1.89	0.70	0.183R
	2.0	1.73	0.72	0.149R
	3.0	1.41	0.79	0.069R
	4.0	1.29	0.83	0.040R
	6.0	1.18	0.88	0.018R
	8.0	1.13	0.91	0.010R
	10.0	1.10	0.92	0.0065R
 8	1.2	3.55	0.67	0.409R
	1.4	2.48	0.72	0.292R
	1.6	2.07	0.76	0.224R
	1.8	1.83	0.78	0.178R
	2.0	1.69	0.80	0.144R
	3.0	1.38	0.86	0.067R
	4.0	1.26	0.89	0.038R
	6.0	1.15	0.92	0.018R
	8.0	1.10	0.94	0.010R
	10.0	1.08	0.95	0.0065R
 For similar sections having same values of K see Table 2 9	1.2	2.52	0.67	0.408R
	1.4	1.90	0.71	0.285R
	1.6	1.63	0.75	0.208R
	1.8	1.50	0.77	0.160R
	2.0	1.41	0.79	0.127R
	3.0	1.23	0.86	0.058R
	4.0	1.16	0.89	0.030R
	6.0	1.10	0.92	0.013R
	8.0	1.07	0.94	0.0076R
	10.0	1.05	0.95	0.0048R
 For similar sections having same values of K see Table 2 10	1.2	2.37	0.73	0.453R
	1.4	1.79	0.77	0.319R
	1.6	1.56	0.79	0.236R
	1.8	1.44	0.81	0.183R
	2.0	1.36	0.83	0.147R
	3.0	1.19	0.88	0.067R
	4.0	1.13	0.91	0.036R
	6.0	1.08	0.94	0.016R
	8.0	1.06	0.95	0.0089R
	10.0	1.05	0.96	0.0057R

* y_o is distance from centroidal axis to neutral axis.

Table 1 (Concluded)
Values of K for Different Sections and Different Radii of Curvature

Section	$\frac{R}{c}$	Factor K		y_o*
		Inside Fiber	Outside Fiber	
 11	1.2	3.28	0.58	0.269 R
	1.4	2.31	0.64	0.182 R
	1.6	1.89	0.68	0.134 R
	1.8	1.70	0.71	0.104 R
	2.0	1.57	0.73	0.083 R
	3.0	1.31	0.81	0.038 R
	4.0	1.21	0.85	0.020 R
	6.0	1.13	0.90	0.0087 R
	8.0	1.10	0.92	0.0049 R
	10.0	1.07	0.93	0.0031 R
 See also I section Table 2 12	1.2	2.63	0.68	0.399 R
	1.4	1.97	0.73	0.280 R
	1.6	1.66	0.76	0.205 R
	1.8	1.51	0.78	0.159 R
	2.0	1.43	0.80	0.127 R
	3.0	1.23	0.86	0.058 R
	4.0	1.15	0.89	0.031 R
	6.0	1.09	0.92	0.014 R
	8.0	1.07	0.94	0.0076 R
	10.0	1.06	0.95	0.0048 R

*y_o is distance from centroidal axis to neutral axis.

varies appreciably for varying section proportions as indicated by the values for the trapezoidal, T, and I sections; but it is felt that the value of the correction factor for any section likely to be used in a curved flexural member may be estimated with sufficient accuracy directly from Table 1.

4. *Empirical Formula for K.*—Although Table 1 may be considered as sufficiently indicating the approximate value of a correction factor for any section used in curved members, nevertheless it seemed desirable to obtain an empirical formula for K on the basis of the values in Table 1, and to check the values obtained by the use of this formula when applied to sections different from those in the table against values calculated in the same way as those in the table. The formula developed is as follows:

$$K = 1.00 + 0.5 \frac{I}{bc^2} \left[\frac{1}{R-c} + \frac{1}{R} \right] \tag{6}$$

(use 1.05 instead of 0.5 for circular and elliptical sections)

I = moment of inertia as used in straight-beam formula

b = maximum breadth of section

c = distance from centroidal axis to inside fiber, i.e., to the extreme fiber nearest the center of curvature

R = radius of curvature of centroidal axis of beam

It will be observed that the value of K in this formula depends on the distribution of the area as involved in the expression $\left(\frac{I}{bc^2}\right)$, and also on the ratio $\left[\frac{1}{R-c}+\frac{1}{R}\right]$ which takes account of the curvature of the section.

In spite of its simplicity this formula for K gives very satisfactory results for all sections (except the triangle) in Table 1 throughout the entire range of curvatures. Values of K obtained by use of this formula will agree with the computed values for any section in Table 1, with the exception of the triangle, with a maximum error of 5 per cent, the average error being considerably less than this. In the case of the triangle the error is as great as 10 per cent in one case, but it is a section rarely used, and was included in the table only as an extreme case of the trapezoid. Hence no particular effort was made to fit its K curve with the formula.

This empirical formula has been tested for numerous section proportions not included in Table 1, and a maximum error of 5 per cent seems to cover all cases. An error of 5 or even 10 per cent, however, is distinctly less than that with which the load on a member can be predicted, or the assumption of homogeneity of material assured. Further, the error in calculating Z by the regular method can easily exceed 10 per cent unless five-place accuracy is used in its determination, and even then the lengthy computation involved makes mistakes readily possible.

The formula as given applies to the inside fiber only, i.e., to the extreme fiber on the concave side. Since the stress in the extreme fiber on the convex side is always less than that given by the straight-beam flexure formula, and since the significant stress is nearly always that at the inside fiber, no attempt has been made to derive an empirical formula from which the K for the outside fiber can be obtained.

It will be observed that the empirical formula satisfies the condition that K approaches unity as the radius of curvature becomes very large, and the member approaches a straight beam. Likewise, as the radius of curvature becomes small (approaches zero) the value of K becomes indefinitely large and hence the calculated stress at the inside fiber becomes infinitely large, which is consistent with the Winkler-Bach formula.

The values of K for some of the sections listed in Table 1 are shown graphically in Fig. 2. The variation in the value of K corresponding to the sharper curvatures is evident.

5. *Limitations and Assumptions.*—It should be noted that the values of correction factors in Table 1 should be used in calculating the

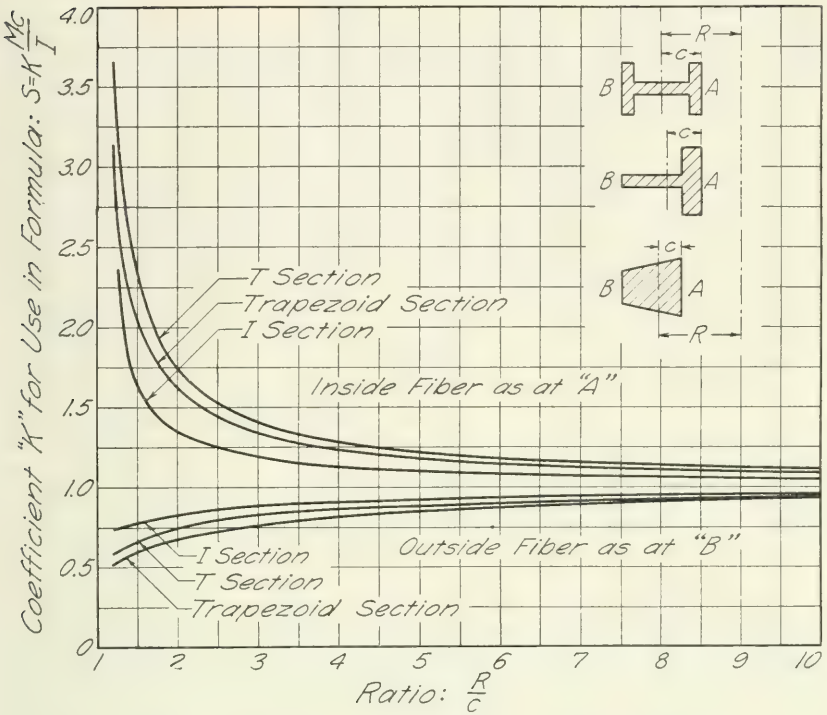


FIG. 2. VARIATION OF K WITH $\frac{R}{c}$ FOR THREE SECTIONS

stress only in the extreme fibers of the curved beam. However, this is where the maximum stress occurs. In order to show the distribution of stress over the cross-section, and to illustrate the fact that the ordinary flexure formula with these correction factors applies only to the extreme fiber, the sketch in Fig. 3 is shown.

One of the fundamental assumptions on which all mathematical determination of stresses in members is based is that the material is homogeneous and isotropic, and hence that there are no discontinuities in the stress distribution on the cross-section. But, although no material satisfies this assumption, the values given by the usual design formulas are in general satisfactory at least for ductile material subjected to static loads. In other words, even though the calculated stress is only an average or statistical value of the stress in the neighborhood of the point at which a knowledge of the stress is desired, it is nevertheless the significant stress for most materials subjected to static loads. Thus the Winkler-Bach curved-beam formula gives the significant stress for most materials. The question is sometimes raised, however, as to whether the value calculated from the curved-beam formula is more significant in the

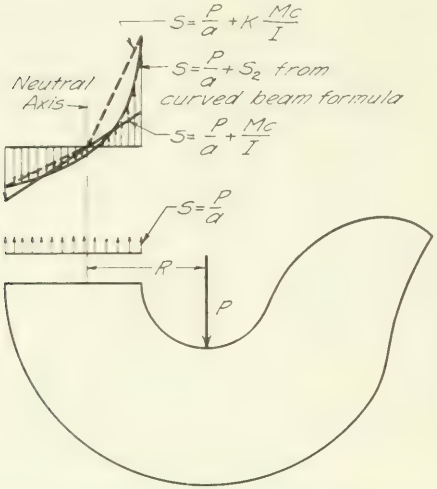


Fig. 3. Stress Distribution in a Curved Flexural Member as Given by Three Formulas

case of cast material, particularly cast iron, than is the value given by the straight-beam formula. In the calculation of the stress in curved beams by means of the uncorrected straight-beam formula, the so-called factor of safety must cover both the uncertainty of the material and the error arising due to the use of an incorrect ormula. (The stress in the curved beam may be more than 3 times as great as the value given by the straight-beam formula, see Table 1). This method of design was perhaps justified because of the difficulties involved in the use of the Winkler-Bach formula. The use of the straight-beam formula with a correction factor which is easily obtained would seem to make available for practical use the advantages of the curved-beam theory, and hence to give a more rational basis for design. Further, this procedure should help to reduce the number of failures of curved flexural members, particularly when such members are subjected to impact and to repeated stress.

The question is also raised as to whether the curved-beam theory should be applied to a curved member at a section where the inner and outer surfaces are not concentric curves. The curved-beam formula is based on the same assumption in this respect as is the straight-beam formula; namely, that the beam has a constant cross-section; or, at least, that the change in cross-section along the beam is very gradual. At the corner sections of some frames where nearly a right angle turn is made, both the straight-beam and the curved-beam formulas would be, per-

haps, considerably in error, but in most cases the value of the stress given by the curved-beam formula would be the better indication of the significant stress.

III. Conclusions

6. *Summary.*—The following brief summary is offered:

(1) The straight-beam flexure formula is radically in error when applied to curved flexural members of relatively sharp curvature, and this error is probably responsible for many failures of curved flexural members particularly when the members are subjected to impact and to repeated stress.

(2) The Winkler-Bach formula for curved beams, although reliable, is difficult of application.

(3) The stress in curved beams can be satisfactorily calculated by the ordinary straight-beam flexure formula used with a correction factor. Values of the correction factor for any sections likely to be used in structural and machine members may easily be obtained by use of Table 1, or equation (6).

(4) The stress computed by means of this corrected straight-beam formula applies only to the extreme fibers of the member. But, since the significant stress usually occurs at an extreme fiber, the method offers a simple, satisfactory, and practical means of designing curved flexural members.

APPENDIX A

Expressions for Section Property Z

As pointed out in the discussion of the Winkler-Bach formula, the evaluation of the integral $Z = -\frac{1}{a}\int \frac{y}{R+y}\,da$ is the chief difficulty encountered in the use of the curved-beam formula. Performing this integration results in complicated expressions for all but the circular and rectangular sections.

By expanding $\frac{y}{R+y}$ into a series we get the general expression

$$Z = \frac{1}{a}\Sigma\left[-\left(\frac{y}{R}\right) + \left(\frac{y}{R}\right)^2 - \left(\frac{y}{R}\right)^3 + \left(\frac{y}{R}\right)^4 \text{ ----}\right]\Delta a$$

For symmetrical sections, the summation of the terms with odd powers is zero and the series converges rapidly. The first three terms are usually sufficient in that case, but where the series expression is satisfactory, the

TABLE 2
ANALYTICAL EXPRESSIONS FOR Z

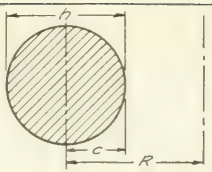
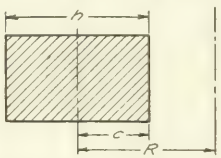
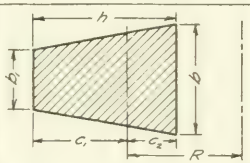
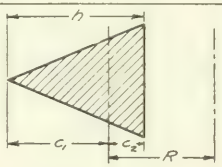
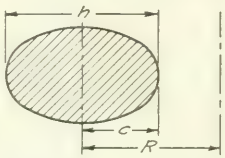
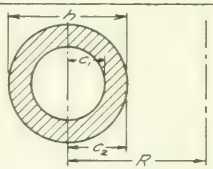
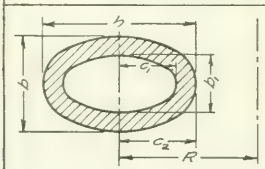
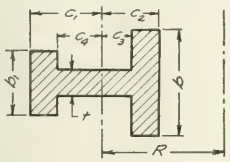
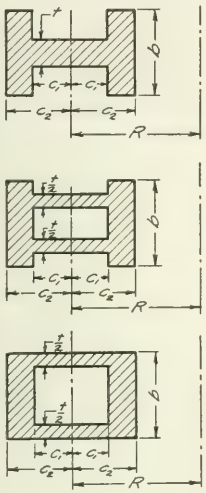
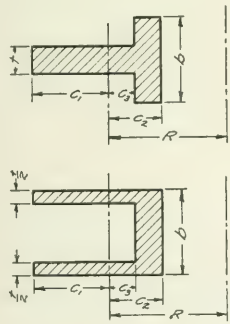
	$Z=\frac{1}{4}\left(\frac{c}{R}\right)^2+\frac{1}{8}\left(\frac{c}{R}\right)^4+\frac{5}{64}\left(\frac{c}{R}\right)^6+\frac{7}{128}\left(\frac{c}{R}\right)^8+\cdots\cdots\cdots$ $Z=-1+2\left(\frac{R}{c}\right)^2-2\left(\frac{R}{c}\right)\sqrt{\left(\frac{R}{c}\right)^2-1}$
	$Z=\frac{1}{3}\left(\frac{c}{R}\right)^2+\frac{1}{5}\left(\frac{c}{R}\right)^4+\frac{1}{7}\left(\frac{c}{R}\right)^6+\cdots\cdots\cdots$ $Z=-1+\frac{R}{h}\left[\log_e\left(\frac{R+c}{R-c}\right)\right]$
	$Z=-1+\frac{R}{ah}\left\{\left[b_1h+(R+c_1)(b-b_1)\right]\log_e\left(\frac{R+c_1}{R-c_2}\right)-(b-b_1)h\right\}$ $Z=-1+\frac{2R}{(b-b_1)h}\left\{b_1+\frac{b-b_1}{h}(R+c_1)\log_e\left(\frac{R+c_1}{R-c_2}\right)-(b-b_1)\right\}$
	$Z=-1+2\frac{R}{h^2}\left[(R+c_1)\log_e\left(\frac{R+c_1}{R-c_2}\right)-h\right]$
	$Z=\frac{1}{4}\left(\frac{c}{R}\right)^2+\frac{1}{8}\left(\frac{c}{R}\right)^4+\frac{5}{64}\left(\frac{c}{R}\right)^6+\frac{7}{128}\left(\frac{c}{R}\right)^8+\cdots\cdots\cdots$ $Z=-1+2\left(\frac{R}{c}\right)^2-2\frac{R}{c}\sqrt{\left(\frac{R}{c}\right)^2-1}$
	$Z=-1+\frac{2R}{c_2^2-c_1^2}\left[\sqrt{R^2-c_1^2}-\sqrt{R^2-c_2^2}\right]$
	$Z=-1+\frac{1}{bc_2-b_1c_1}\left\{bc_2\left[2\left(\frac{R}{c_2}\right)^2-2\left(\frac{R}{c_2}\right)\sqrt{\left(\frac{R}{c_2}\right)^2-1}\right]\right.$ $\left.-b_1c_1\left[2\left(\frac{R}{c_1}\right)^2-2\left(\frac{R}{c_1}\right)\sqrt{\left(\frac{R}{c_1}\right)^2-1}\right]\right\}$

TABLE 2 (Concluded)
ANALYTICAL EXPRESSIONS FOR Z

	$Z = -1 + \frac{R}{a}\left[b_1 \log_e (R+c_1) + (t-b_1)\log_e (R+c_4) + (b-t)\log_e (R-c_3) - b\log_e (R-c_2)\right]$
	The value of Z for each of these three sections may be found from the expression above by making: $b_1 = b$, $c_2 = c_1$, and $c_3 = c_4$ $Z = -1 + \frac{R}{a}\left[b\log_e\left(\frac{R+c_2}{R-c_2}\right) + (t-b)\log_e\left(\frac{R+c_2}{R-c_2}\right)\right]$ $Area = a = 2\left[(t-b)c_1 + bc_2\right]$
	In the expression for the unequal I given above make $c_4 = c_1$ and $b_1 = t$, then $Z = -1 + \frac{R}{a}\left[t\log_e (R+c_1) + (b-t)\log_e (R-c_3) - b\log_e (R-c_2)\right]$ $Area = a = tc_1 - (b-t)c_3 + bc_2$

exact evaluation of the integral is also comparatively simple, and is, of course, more accurate. For the unsymmetrical sections the series converges too slowly to be of use.

In order that the expressions for Z used in calculating the values in Table 1 may be made available for those wishing to use the curved-beam formula, Table 2 is presented. The evaluated integral is given for all

sections commonly found in curved flexural members. The expressions were obtained from Bach's, "Elastizität und Festigkeit," and from an unpublished paper by A. E. Hershey, and have been revised to fit the notation of the figures.

APPENDIX B

Graphical Determination of Z by Line Polygon Method

Reference has been made to the "line polygon" method of evaluating Z graphically. This method, devised by A. E. Hershey, is slightly more accurate than the "link rigidity" method, and does not necessitate the use of a planimeter or other area measuring device. It is particularly convenient when the string-polygon method is used in locating the centroidal axis of the section, and a brief description of it will be given here.

In Fig. 4, the section whose Z is to be determined is enclosed by the heavy-line curve $EABF$, etc. Divide the area into several strips of equal width (about ten strips should be used for accurate results). Draw the mean ordinates AB, CD, EF, etc., of the strips. Then lay off the mean ordinates on a line $A'B'C'D'$, etc. The mean ordinate of any strip being assumed proportional to the area of the strip, the total length of this line then represents the area of the section

If the centroidal axis is to be found graphically, select a pole J and draw the rays JA', JB', etc. Draw aP parallel to JA', and starting at a, draw the strings ax_1, x_1x_2, etc., corresponding to the rays JB', JC', etc., to locate x. Then draw xP parallel to JK', thus locating P. The line MM' through P perpendicular to the center line of the section is the centroidal axis, and M is the centroid of the area.

Now to obtain a value of Z, draw a line RS parallel to OM and at a distance of unity from OM, using the same scale as used in drawing the section. From the intersections V_1, V_2, etc., of RS with the median lines, draw lines to the center of curvature O. These lines locate points Q_1, Q_2, etc., where they intersect the centroidal axis. Now, at the ends of the vectors $A'B'$, $B'C'$, etc., in the vector polygon, erect perpendiculars $B'b$, $C'c$, etc., the lengths of which are unity to the scale used in drawing the vector polygon. From the ends of these lines lay off lengths bb', cc', etc., equal to MQ_1, MQ_2, etc., respectively. Through each of the points b', c', d', etc., thus obtained, draw lines q_1, q_2, q_3, etc., parallel to the vector polygon $A'B'C'D'E'F'G'H'K'$. The value of Z is obtained by dividing

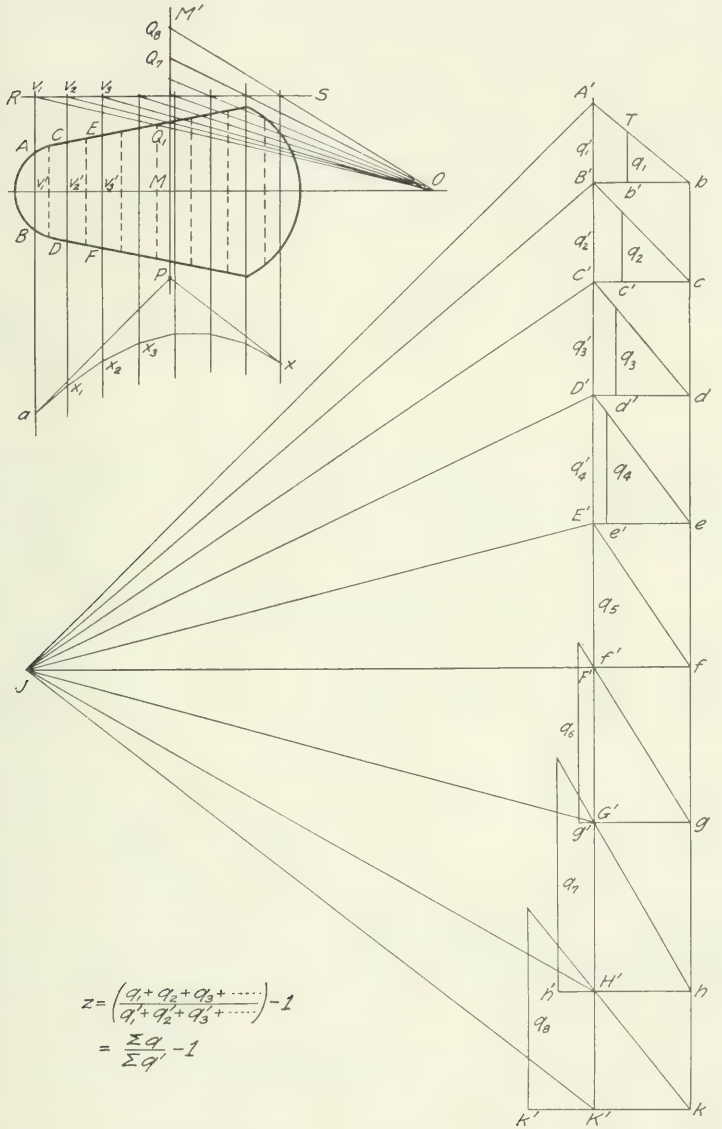


FIG. 4. GRAPHICAL DETERMINATION OF Z BY THE LINE POLYGON METHOD

the sum of the lengths q_1, q_2, q_3, etc., by the area of the section (represented by the sum of the lengths q_1', q_2', q_3', - - - - - -), and subtracting unity from the quotient. That is

$$Z = \frac{\Sigma q}{\Sigma q'} - 1$$

Proof: By definition

$$Z = -\frac{1}{a}\int \frac{y}{y+R}\,da$$

This may be written

$$Z = -\frac{1}{a}\int\left(1 - \frac{R}{y+R}\right)da = -\frac{1}{a}\int da + \frac{1}{a}\int \frac{R}{y+R}\,da$$

$$= -1 + \frac{1}{a}\int \frac{R}{y+R}\,da$$

Now each strip is an elementary area da and is at a distance y from the centroidal axis. That is, $MV_1' = y$; also $OM = R$. Then in triangles $OV_1'V_1$ and OMQ_1 we have

$$\frac{MQ_1}{V_1'V_1} = \frac{OM}{OV_1'}, \text{ or } MQ_1 = \frac{R}{y+R} \text{ since } V_1'V_1 = 1$$

In the triangle $A'B'b$, $\frac{b'T}{b'b} = \frac{A'B'}{B'b}$ but $b'b = MQ_1 = \frac{R}{y+R}$

$B'b = 1$, and $A'B' = da$. Therefore $b_1T = q_1 = \frac{R}{y+R}\,da$

Hence $q_1 + q_2 + q_3 + \text{ - - - - - } = \Sigma q = \int \frac{R}{y+R}\,da$

Also, area $a = q_1' + q_2' + q_3' + \text{ - - - - - } = \Sigma q'$

Therefore $Z = \frac{\Sigma q}{\Sigma q'} - 1$

APPENDIX C

Derivation of the Winkler-Bach Formula*

Let the curved beam DOE (Fig. 5a) be subjected to the loads and reactions Q.

Let AB and A_1B_1 be two normal sections of the beam indefinitely close together before the loads are applied. After the beam is loaded the

*The theory here given is substantially that laid down by Bach, "Elastizität und Festigkeit," Sec. 54.

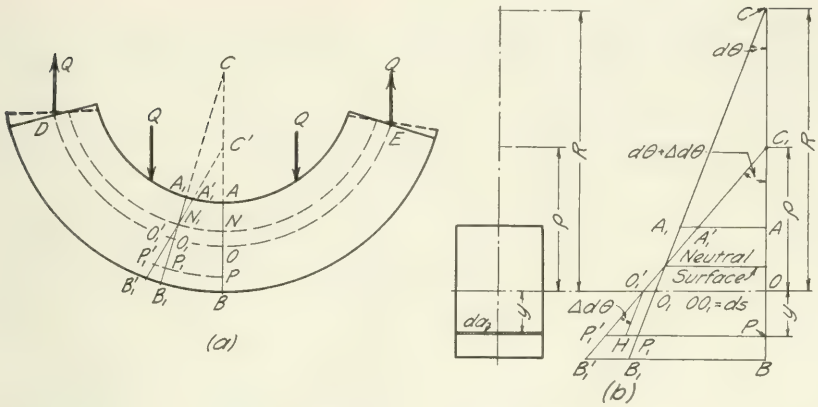


Fig. 5. Strains in a Curved Beam

change in length of any fiber PP_1 is assumed to be proportional to its distance from the neutral axis and is represented by P_1P_1'; the neutral surface is represented by NN_1. It is required to find the unit stress on any point P in a cross-section of a curved beam in terms of the bending moment and the dimensions of the section.

Let Fig. 5b represent the part ABB_1A_1 of Fig. 5a enlarged. The unit stress S on a fiber PP_1 (Fig. 5b) at any distance y from the centroidal axis is

$$S = Ee \tag{1}$$

in which E is the modulus of elasticity of the material and e is the unit strain of the fiber

The bending moment M causes the plane A_1B_1 to rotate through an angle $\Delta d\theta$ thereby changing the angle this plane makes with the plane BAC from $d\theta$ to $(d\theta + \Delta d\theta)$; the center of curvature is changed from C to C', and the distance of the centroidal axis from the center of curvature is changed from R to ρ. It should be noted that y, R, and ρ at any section are measured from the centroidal axis, and not from the neutral axis.

Let ds denote the length of the centroidal fiber OO' and e_0 the unit strain of this fiber. Then

$$e_0 = \frac{O_1O_1'}{OO_1} \text{ or } O_1O_1' = e_0 ds = e_0 R d\theta$$

$$e = \frac{P_1P_1'}{PP_1} = \frac{P_1H + HP_1'}{PP_1} = \frac{O_1O_1' + HP_1'}{PP_1}$$

But, as found above, $O_1O_1' = e_0 R d\theta$;

further $HP_1' = O_1'H\ \Delta d\theta = y\Delta d\theta$;

also $PP_1 = (R + y)\ d\theta$

$$\text{Therefore } e = \frac{e_0 R d\theta + y\Delta d\theta}{(R + y)\ d\theta}$$

$$= \frac{e_0 R + y\ \dfrac{\Delta d\theta}{d\theta}}{R + y}$$

For convenience let the angular unit strain $\dfrac{\Delta d\theta}{d\theta}$ be denoted by ω. By adding and subtracting $e_0 y_1$ the expression for e may be transformed to the following equation:

$$e = e_0 + (\omega - e_0)\ \frac{y}{R + y} \tag{2}$$

From equation (1) then

$$S = Ee = E\left[e_0 + (\omega - e_0)\ \frac{y}{R + y}\right] \tag{3}$$

Since there are three unknown quantities (S, ω, and e_0) in this equation, two additional equations must be found. These are obtained from the conditions of equilibrium: namely, (1) that the algebraic sum of the external forces about any axis in the cross-section is equal to the moment of the stresses on the section about the same axis, that is, the resisting moment is equal to the bending moment, or $\int syda = M$; and (2) that the algebraic sum of all the normal stresses on the cross-section is equal to zero, or $\int sda = 0$.

Thus

$$M = \int ysda = \int Ey\left[e_0 + (\omega - e_0)\ \frac{y}{R+y}\right] da \tag{4}$$

$$\int sda = \int E\left[e_0 + (\omega - e_0)\ \frac{y}{R+y}\right] da = 0 \tag{5}$$

If the modulus of elasticity E has the same value for all the fibers, it is a constant in the equations, and hence equations (4) and (5) may be written

$$M = E\left[e_0 \int y da + (\omega - e_0) \int \frac{y^2}{R+y}\, da \right] \tag{6}$$

$$\text{and } e_0 \int da = -(\omega - e_0) \int \frac{y}{R+y}\, da \tag{7}$$

But $\int da = a$, and since y is measured from the centroidal axis, $\int y da = 0$. Further, for convenience, let

$$\int \frac{y}{R+y}\, da = -Za \tag{8}$$

in which Z is a property of the area somewhat similar to the moment of inertia in the straight-beam formula. The integral in equation (6) may also be expressed in terms of Z as follows:

$$\int \frac{y^2}{R+y}\, da = \int \left(y - \frac{Ry}{R+y} \right) da = -R \int \frac{y}{R+y}\, da = ZaR$$

Equations (6) and (7) now become

$$M = E\ (\omega - e_0)\ ZaR \tag{9}$$

$$e_0 = (\omega - e_0)\ Z \tag{10}$$

$$\text{Hence } (\omega - e_0) = \frac{M}{EZaR},\ e_0 = \frac{M}{EaR} \text{ and } \omega = \frac{1}{Ea}\left(\frac{M}{R} + \frac{M}{RZ} \right) \tag{11}$$

Substituting these values in equation (3)

$$S = \frac{M}{aR}\left[1 + \frac{1}{Z}\left(\frac{y}{R+y} \right) \right] \tag{12}$$

in which S is the unit stress at a point at the distance y from the centroidal axis of the section (y becomes c at the extreme fibers); M is the bending moment at the given section; R is the radius of curvature of the centroidal axis of the section of the unstressed beam; a is the area of the cross-section; and Z is a property of the cross-section defined above.

The bending moment M is positive when it increases curvature and negative when it decreases curvature; y is positive when measured toward the convex side of the beam and negative when measured toward the concave side, that is, toward the center of curvature.

Location of Neutral Axis

According to equation (12) the unit stress on any fiber at the distance y from the centroidal axis, due to the bending moment M, is

$$S = \frac{M}{aR}\left[1 + \frac{1}{Z}\left(\frac{y}{R+y}\right)\right] \tag{13}$$

The stress at the neutral axis is zero, and if y_0 denotes the distance of the neutral axis from the centroidal axis, the value of y_0 may be found by equating S in the above equation to zero.

Thus

$$1 + \frac{1}{Z}\left(\frac{y_0}{R+y_0}\right) = 0$$

$$y_0 = -\frac{ZR}{Z+1} \tag{14}$$

The negative sign means that y_0 is measured from the centroidal axis toward the center of curvature. It is important to note that equation (14) applies to a beam that is subjected to bending only.

Axial and Bending Loads Combined

If a curved beam is subjected to a normal load P acting through the centroid of the area of the section (axial load) in addition to a bending moment, the resulting unit stress is the algebraic sum of the unit stresses caused by the axial and the bending loads. Thus the maximum unit stress is

$$S = \frac{P}{a} \pm \frac{M}{aR}\left[1 + \frac{1}{Z}\left(\frac{y}{R+y}\right)\right] \tag{15}$$

the signs of M and y are determined as stated above, and the sign of P is positive when it produces tensile and negative when it produces compressive stress.

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